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"Electrified Pate"

EVERYTHING AND MORE: A COMPACT HISTORY or ∞
By David Foster Wallace. W, W. Norton and James Atlas Books, $23.95.
Reviewed by JOHN ALLEN PAULOS

In his oxymoronically titled *Everything and More*, David Foster Wallace sketches the history of humanity's attempts to understand the notion of infinity. His book begins with the Greeks, Zeno in particular, and ends with modern logicians, Georg Cantor in particular. In between are the mathematicians who used calculus, invented in the seventeenth century by Newton and Leibniz, to try to get a handle on the discombobulating notions of the infinitely big and the infinitesimally small.

Wallace is best known as the author of the boundlessly exuberant 1.088-page novel *Infinite Jest*, among other works of fiction. He brings to his task a refreshingly conversational style as well as a surprisingly authoritative command of mathematics. The book is chock-full of detail, unsoftened by the personality-mongering and scene-setting common in most popular science writing. Because the language is smart and inventive, however, the book provides enough enjoyment to induce the mathematically unsophisticated reader to slog through the many difficult patches along the way. But while most people will certainly get something out of Everything, it's hard to imagine that even a reader with a couple of years of college math could fully understand the mathematical and historical minutiae.

And the philosophical material that. Wallace addresses is even thornier than the mathematics. In approximately 450 B.C., the Greek philosopher Zeno posed some troubling puzzles that, in Wallace's words, "pulled the starter rope on everything." The most famous of these paradoxes, known as the Dichotomy, asks how we can ever cross a road, (Another version of the paradox concerns the impossibility of Achilles' ever catching die tortoise.) To cross a road, Zeno argues, we must first go halfway across. To do this, however, requires that we first go one-quarter of the way across, which requires that we first go one-eighth of the way across, and so on forever. We can't cross the road without completing all of these infinitely many acts. But since traversing each of these distances takes some time, traversing an infinity of them must take an infinite amount of time. (This brings to mind a Zenonian variant of an old chestnut: Why didn't the chicken cross the road?)

Zeno also raised the problem of infinity in other "crunchers," as Wallace calls them. In the paradox of the arrow, Zeno claims that an arrow stands still even when it is in midflight. At any particular instant, the arrow simply is where it is and occupies a volume of space exactly equal to itself. During this instant the arrow cannot move, since, if it did, at least one of two ludicrous consequences would follow. Either the instant would have to contain an earlier and a later part-and instants by definition don't have parts-or else the arrow would have to occupy a volume of space larger than itself, in order for it to have room to change position. Neither of these ideas makes sense, so we must conclude that motion during an instant, and hence motion itself, is impossible.

Zeno's conclusions are clearly false, but what exactly is wrong with his arguments? In treating these and other paradoxes, Wallace refrains from quickly and glibly citing the modern mathematical techniques developed to deal with them, and instead stresses the truly mind-boggling nature of these lacunae in our intuition. This is a welcome approach, and it reflects the actual historical development of mathematics. Too often in mathematical pedagogy, the intimate relationship between mathematics and philosophy, literature, history, science, and everyday life is ignored or trivialized.

What are those modern resolutions to the f problem of infinity? The answer is long, and Wallace painstakingly leads the reader through its many stages: the invention and refinement of calculus, limiting processes, infinite series, epsilon-delta proofs, Cauchy sequences, Dedekind's definition of real numbers, Weierstrass and the arithmetization of analysis, uniform convergence, Fourier series, and the topology of the real number line. The "hero" of the saga is die late-nineteenth-century German mathematician Georg Cantor, the founder of modern set theory, whose work revolutionized mathematics. Insisting on the actual, rather than the merely potential, reality of infinity, Cantor came to the subject not through abstract speculation, but through the nitty-gritty problems of modern mathematical analysis.

Cantor eventually did arrive at a general definition of infinity, based on mathematical sets, and it is this idea that Wallace stresses. Take, for example, the counterintuitive notion of an infinite set. A set is infinite if it can always be put in one-to-one correspondence with a subset of itself; that is, it has a subset whose members can be paired up one-to-one with the members of the whole set. Thus the set of whole numbers is infinite, since we can pair its members with a subset-for example, the multiples of 13: 1-13, 2-26, 3-39, 4-52, 5-65, 6-78, and so on ad infnitum. Two sets are said to contain the same number of members if there is such a one-to-one correspondence between them; strangely, then, there are just as many multiples of 13 as there are whole numbers.

As Wallace notes, some of the oddities associated with infinite sets had been known since Galileo. But only because of Cantor's systematic efforts did set theory become the common language and foundation of mathematics. One Cantorian idea, in particular, was fundamental: the distinction between countably infinite and uncountably infinite sets. A set is countably infinite if there is some way to match up its elements in a one-to-one fashion with the positive whole numbers. An un-countably infinite set is one whose elements cannot be so matched up. The whole numbers are countably infinite, and, as Cantor showed in a classic proof that Wallace describes, so are the rational numbers (all fractions).

But is there an infinite set that is "more infinite," that can't be put into a one-to-one correspondence with the whole numbers? Cantor showed that there is. The set of all real numbers (all decimals) is more numerous than the set of integers or the set of rational numbers, and thus has a higher "cardinality." In other words, there isn't any way to match up the real numbers with the integers in a one-to-one fashion without some real numbers being left over. Wallace also rehearses the standard proof of this fact, which is indirect and beautiful, but which, alas, can't be fit into this brief review.

There are sets that have even higher cardinalities than the set of real numbers. In fact, a whole hierarchy of infinite cardinalities exists. Cantor speculated, however, that there was no subset of real numbers that had a cardinality greater than that of the whole numbers, yet smaller than that of the real numbers. This speculation has come to be called the continuum hypothesis, and, despite breaking his head against it. Cantor never proved it. (The Austrian logician Kurt Godel and the American mathematician Paul Co-hen later showed why.)

Wallace touches on all these matters and many more, from the ancient Greeks' consternation at the discovery of irrational numbers all the way through Abraham Robinson's development, in the 1960s, of non-standard analysis as a possible way to deal formally with infinitesimals. (And he frequently cites his own beloved high school math teacher, Dr. Goris.) As this suggests. Everything and More is absurdly ambitious. But it achieves enough of its transcendent goals to be judged a success. It makes us more keenly aware of how astonishing it is that our "2.8 pounds of electrified pate," as Wallace describes our brains, can comprehend, imagine, or at least deal with, the infinite.

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